Comparison of Antractic Surface Area and Coastline Length in the 2015 Glacial Cycle with Regard to Fractal Dimension

International Baccalaureate Diploma Programme, November 2022

Mathematics HL Exploration

Contents

1. Introduction

1.1 Background Information

The word "fractal," which means broken or fractured in Latin *frāctus*, was first used by the mathematician Benoit Mandelbrot in the 20th century. Fractal geometry, like any other geometry, has existed in nature before the term was even coined. Examples include tree branches, river networks, snowflakes, and coastlines. A subset of Euclidean geometry known as fractal geometry is characterized by having the same appearance at all scales. Therefore, theoretically, at smaller scales, the shape would include infinitely repeating patterns; this is known as "self-similarity." Therefore, if we zoom in on a fractal, we will see an exact replica of it. However, in nature, fractals do not have the same appearance at all scales, and this phenomenon is known as "not scale-free." It may appear that a coastline or a land frontier repeats the same pattern, but this is untrue. It indicates that there isn't a consistent pattern along the entire coastline. The relationship between fractals and coastlines is described in an article by Benoit Mandelbrot titled "How Long Is the Coast of Britain?" (Mandelbrot, 1967). He claims that although measuring a coastline's length does not have a fixed value, we can measure its fractal dimension objectively.

1.2 Aim

The aim of this exploration is to look at Great Britain's fractal dimension using the self-similarity (D_S), the Hausdorff-Besicovitch (D_H), the Minkowski-Bouligand (D_B) dimension and area-perimeter realationship with fractal dimension will all be used. Then for each method, the percent error is going to be determined. The changes in Antarctica's surface area and coastline length during the 2015 glacial cycle will then be determined as a case study utilizing the methodologies that are being studied. In this manner, the fractal dimension is used to calculate the challenges that we will encounter as a result of climate change.

2. Fractals and the Coastline Paradox

A coastline's "length" has no bearing on geographical definition. because it can be changed depending on the map's scale. When we look at coastlines broadly, we can draw curves and measure them, but when we look at them closely, we see many large and small rocks, making it nearly impossible to measure their perimeters. This is known as the “coastline paradox” in mathematics. Figure 1's maps are arranged from a broad view to a more detailed one. As a result, the length of a coastline is determined by the map's scale (Falconer, 1990).

Figure 1 *A map of Great Britain that includes close-up images of its coastline (Google Earth).*

2.1 Dimension

According to the Oxford Dictionary, a dimension is "a measurement in space." Astronomer Dave Kornreich defined a manifold's dimension as the smallest integer number of coordinates required to uniquely identify each point in that manifold (Kornreich, 2015). Here, the term "manifold" is used to refer to dimensions generally. Fractal shapes don't have integer dimension values, which is the knowledge that this investigation is centered on.

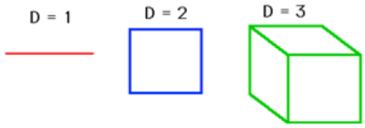


Figure 2 Displays the shapes with various dimensions.

2.2 Fractal Dimension

Fractal dimension is a metric used to assess the complexity of a pattern. Besides the length of a coastline or a land border, a fractal dimension has an objective result. The fractal dimension of Great Britain has already been identified by various sources. All of the outcomes are comparable, while not being exactly the same. The calculations used several methods for fractal dimension determination, which is why the outcomes were inconsistent. The study of Benoit Mandelbrot is the one that is most frequently quoted in literature when discussing the fractal dimension of Great Britain (Husain et al., 2021). The value will be used to compare the measured fractal dimension, and the result is listed in Table 1 according to them (Benoit Mandelbrot, 2006).

Table 1 *Fractal dimension of Great Britain's coastline according to resources.*

Coastline	Fractal Dimension (D_M)
Great Britain	1.25

Fractal's dimension falls between 1 and 3. At all natural scales, fractals do not consistently appear. Since they are not self-similar, the self-similarity method cannot be used to find the fractal dimension of coastlines. The fractal dimensions of fractals found in nature are typically calculated using measurable dimensions, such as the Hausdorff and Minkowski Bouligand dimensions. The following equation relates the measurable dimensions (D) to the Mandelbrot's fractal dimension (D_M):

$$D_M = 1 + |D| \quad (1)$$

The only way to obtain a dimension value in a measurable dimension is to rewrite the equation in linear form. The following is the linear form:

$$y = mx + b \quad (2)$$

As the power law states, exponential terms are written in coefficient form when rewriting an equation with them. In equations with measurable dimensions, the dimension value is typically included in the exponential part (Bovill, 2013). In the next pages, the application of this theoretical information will be shown.

3. Fractal Dimension Determination Methods

This study utilizes the self-similarity, Hausdorff-Besicovitch, Minkowski-Bouligand, and area-perimeter ratio as fractal dimension determination methods (Frame et al., 2016). The fractal dimension of Great Britain will be calculated using the other three methods. Finally, the percent error for each method will be calculated.

3.1 The Self-Similarity Dimension (D_s)

The dimension of a self-similar pattern is determined by the number of smaller sections that the original pattern is divided into and the scaling factor. According to power law, this relationship is represented as follows:

$$a = \frac{1}{(s)^{D_s}} = \left(\frac{1}{s}\right)^{D_s} \quad (3)$$

Where,

D_s = Self-similarity dimension

a = Number of smaller pieces

s = Scaling factor

The given equation can be adjusted using logarithmic rules to find the value of D_s:

$$a = \frac{1}{(s)^{D_s}} = \left(\frac{1}{s}\right)^{D_s} \quad (4)$$

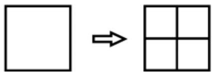
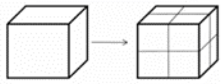
$$\log a = \log \frac{1}{(s)^{D_s}} = \log \left(\frac{1}{s}\right)^{D_s} \quad (5)$$

$$\log a = D_s \log \left(\frac{1}{s}\right) \quad (6)$$

$$D_s = \frac{\log a}{\log\left(\frac{1}{s}\right)} \tag{7}$$

Fractal dimensions are not integers, as was previously mentioned. The dimension of nonfractal structures is also determined by the self-similar dimension, which is not only used for fractals. I would look more closely at the dimensions of a line, a square, and a cube to better comprehend the term "dimension." Their dimension values will be determined using the self-similarity dimension (Bovill, 2013).

Table 2 *Listed three manifolds' dimensions in literature, then Equation 7 is used to find the self-similarity dimension.*

			
D in literature	1	2	3
a	2	4	8
s	$1/2$	$1/4$	$1/8$
D_s	1	2	3

3.2 The Hausdorff-Besicovitch Method (D_H)

The Hausdorff-Besicovitch method is used to calculate a shape's fractal dimension. This technique is typically applied to shapes that are not self-similar. This technique reduces the size of the straight-line segments surrounding the form. In this manner, the design's details are also taken into account, resulting in a longer overall perimeter for the shape as the length of a straight-line segment decreases (Heinz-Otto Peitgen et al., 2012).

This method will be used to determine the fractal dimension of a coastline in this exploration. A coastline is created by bays and ridged inlets. This coastline's complexity and roughness are due to a determined fractal dimension. The link between the frarctal dimension and the length of the coastline is given by the equation that follows, which is stated by Mandelbrot:

$$L = G^{(1-D_H)}M \tag{8}$$

Where,

L = Coastline length

G = Straight-line length

D_H = The Hausdorff-Besicovitch fractal dimension

M = Proportionality constant

The equation was rewritten in log form to obtain the equation in linear form Equation (2).

$$\log(L) = \log{(G)}^{1-D_H} M \quad (9)$$

$$\log(L) = \log{(G)}^{1-D_H} + \log(M) \quad (10)$$

$$\log(L) = (1 - D_H)\log{(G)} + \log(M) \quad (11)$$

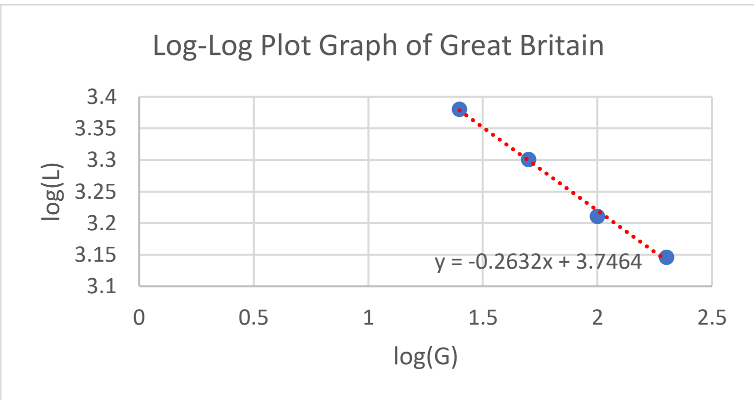
Equation (11), is appropriate for a log-log plot graph which has a slope of $(1 - D_H)$. After the collected data has been graphed, the fractal dimension of the form will be estimated using the gradient of the graph. The next section will examine a fractal dimension using the length of the British coastline at a fixed scale.

On the map of Great Britain in Table 3, various lines have already drawn at various proportionality constants. The coastline of Great Britain appears to be increasing in accuracy as straight-line segments get shorter. n is the number of straight lines utilized in the table. As a result, multiplying the quantity of straight lines (n) by the length of each straight line (G) yields the coastline's perimeter (L).

Table 3 *Great Britain, The Hausdorff-Besicovitch Dimension method with different unit lengths (Bovill, 2013).*

				
L (mi)	1400 miles	1625 miles	2000 miles	2400 miles
G (mi)	200 miles	100 miles	50 miles	25 miles
n	7	16.25	40	96
$log(L)$	3.146128036	3.210853365	3.301029996	3.380211242
log (G)	2.301029996	2	1.698970004	1.397940009

The data will now be plotted in a log-log format. The fractal dimension of Great Britain will be represented by the line graph's gradient.



Graph 1 *Great Britain's The Hausdorff-Besicovitch fractal dimension as a log-log plot graph.*

The data from Table 3 are displayed in blue plots in Graph 1. The red dotted line was made to represent the gradient's average value. In this situation, the gradient is "increase over run," as seen on graph "-0.2632." When recalled, Equation (1) will be applied to transform the Hausdorff-Besicovitch dimension into the Mandelbrot dimension:

$$D_M = 1 + |(-0.2632)| \tag{12}$$

$$D_M = 1.2632 \tag{13}$$

The fractal dimension of Great Britain is "1.25", according to Table 1 in the literature. The gathered data will now be compared to the literature, and the percent error will be calculated.

$$\delta = \frac{|V_A - V_E|}{|V_E|} \times 100 \tag{14}$$

Where,

δ = Percent Error

V_A = Approximate (measured) Value

V_E = Exact Value

To determine the percent error for fractal dimension and slope, using Equation (14):

$$\delta = \frac{|1.2632 - 1.25|}{|1.25|} \times 100 \tag{15}$$

$$\delta = \%1.06 \tag{16}$$

3.3 The Minkowski-Bouligand Method (D_B)

The box-counting method is another name for this methodology. According to literature sources, this methodology is the most trustworthy one (Jens Feder, 1988). This method's equation:

$$N = M({}^{1}/_{\varepsilon})^{D_B} \tag{17}$$

Where,

N = Number of boxes

ε = Boxes' side length

M = Proportionality constant

D_B = The Minkowski-Bouligand fractal dimension

Equation (17) is rearranged in log form as follows to produce a linear equation:

$$\log(N) = D_B \log({}^{1}/_{\varepsilon}) + \log(M) \tag{18}$$

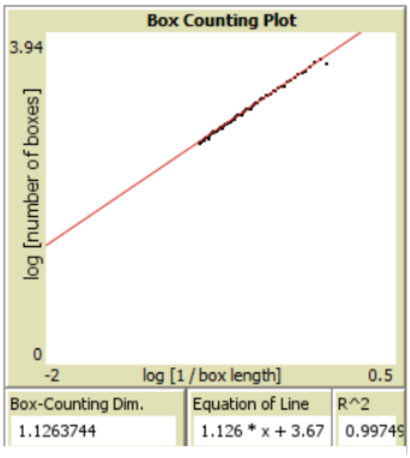
The Minkowski-Bouligand procedure resembles a fundamental calculus approach. On the curve whose fractal dimension will be examined, a grip of square boxes is first placed. The boxes containing the curves are then counted. When the grid's squares get smaller, this process is repeated. Similar to limiting to zero in calculus:

$$D_B = \lim_{\varepsilon \to 0} \frac{\log\left(\frac{N}{M}\right)}{\log\left(\frac{1}{\varepsilon}\right)} \tag{19}$$

Finding results using this "box-counting" method typically takes a long time. At this point, a program written in NetLogo by Melanie Mitchell will be used (Mitchell, n.d.). As a program parameter, the starting box-length is set to 1, and the increment is set to 0.1, allowing for a difference between the data plotted in various box sizes. This value was the minimum to use on the program. In this way, the result is trying to get more accurate. Once the software has run for a time, Table 2 and Graph 2 are generated. Out of 100 iterations, only four of them are listed in Table 2. In Graph 2, a point is set for each iteration, after the 100th iteration, the plot becomes non-linear, most presumably due to systematic software error.

Table 2 *To determine Great Britain's fractal dimension, the Minkowski-Bouligand method is used with NetLogo software.*

				
N	920	1396	2095	3832
ε	4	3	2	1
Iteration #	30	20	10	1



Graph 2 *Obtained from the NetLogo program, the Minkowski-Bouligand log-log plot graph and its gradient value (fractal dimension) for Great Britain.*

The discovered Minkowski-Bouligand fractal dimension is 1.1263744, as seen in Graph 2. The values will now be compared against the literature, after which the error percentage will be calculated. Equation (14) will be used to calculate the percent error:

$$\delta_B = \frac{|1.1263744 - 1.25|}{|1.25|} \times 100 \tag{20}$$

$$\delta_B = \%9.89 \tag{21}$$

3.4 The Surface Area to Perimeter Ratio of Fractal Structures

The following describes the relationship between a fractal surface's surface area and perimeter at the same dimension:

$$\rho = \frac{L}{A^{1/2}} \tag{22}$$

Where,

ρ = The ratio that corresponds to all closed curves of the same shape

A = Surface area

L = The area's perimeter

To keep the same ratio after rearrangement at different dimensions:

$$\rho = \frac{L^{1/D}}{A^{1/2}} \tag{23}$$

This equation does not take into account the straight-line length, which I use to determine the fractal dimension. In a resource this equation has already revised to use this parameter directly with the proportionality constant (M) and straight-line length (G) (Chen, 2013):

$$L = G^{(1-D_H)}M \tag{24}$$

$$\frac{L}{G} = M\left(\frac{A}{G^2}\right)^{D/2} \tag{25}$$

Equation (25) has now been rearranged using logarithmic laws to isolate the D value:

$$log\frac{L}{G} = \log(M) + \frac{D}{2}\left(log\frac{A}{G^2}\right) \tag{26}$$

$$\begin{cases} log\dfrac{L_1}{G_1} = \log(M) + \dfrac{D}{2}\left(log\dfrac{A_1}{{G_1}^2}\right) \\ log\dfrac{L_2}{G_2} = \log(M) + \dfrac{D}{2}\left(log\dfrac{A_2}{{G_2}^2}\right) \end{cases} \tag{27}$$

$$log\frac{L_2}{G_2} - log\frac{L_1}{G_1} = \frac{D}{2}\left(log\frac{A_2}{{G_2}^2} - log\frac{A_1}{{G_1}^2}\right) \tag{28}$$

$$D = \frac{2\log\left(\frac{L_2G_1}{L_1G_2}\right)}{\log\left(\frac{A_2{G_1}^2}{A_1{G_2}^2}\right)} \tag{29}$$

The area-perimeter ratio method uses the Hausdorff-Besicovitch method to calculate the perimeter value that is the most reliable with a %1.06 error. With this approach, the surface area will be calculated using the Green's Theorem in order to obtain the fractal dimension.

The relationship between the circulation around a closed area R and the curl of the vector field inside R is defined by the Green's Theorem. In essence, it says that the total of the curls over a closed area R is equal to the counterclockwise circulation around R. $\oint$ written as: when taken as a line integral over a closed curve R (Gregory Neil Hartman et al., 2015). The Green's Theorem equation is as follows:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R curl\vec{F} dA \tag{30}$$

Where,

A = Area of Surface

R = Plane region with a closed boundary

C = A boundary made up of a finite number of smooth curves.

$\vec{r}$ = Counterclockwise parametrization of C

$\vec{F}$ = A vector field, $\langle M, N \rangle$ where M_x and N_y are partial derivatives and contiuous over R.

By integrating along the boundary of an enclosed region, one may apply the Green's Theorem to determine its area. The equation demonstrates how the "circulation" of F around C is altered when the two-dimensional graph's $d\vec{r}$ is rewritten as dx and dy and the vector field $\vec{F}$ is set to $\langle M, N \rangle$. The equation that follows is written down after Equation (30) has been rearranged:

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial \mathrm{x}} - \frac{\partial M}{\partial \mathrm{y}} \right) dA \tag{31}$$

The double integral $\iint_R dA$, with 1 as the integrand, is known to be used to calculate the area of R. Then surface area can be determined by creating a field where $curl\vec{F} = 1$. If integrand in Equation (31) $\frac{\partial N}{\partial \mathrm{x}} - \frac{\partial M}{\partial \mathrm{y}}$ would equals to 1, it would be possible to determine the area of R with $\oint_C \vec{F} \cdot d\vec{r}$, using the Green's Theorem. For instance, if these partial derivatives are determined to be $N(x, y) = \frac{1}{2}x$ and $M(x, y) = \frac{-1}{2}y$, in this way, $\frac{\partial N}{\partial \mathrm{x}} - \frac{\partial M}{\partial \mathrm{y}}$ will be equal to 1 and values are placed into Equation (31):

$$\oint_C \frac{1}{2}xdy - \frac{1}{2}ydx = \iint_R \left(\frac{\partial N}{\partial \mathrm{x}} - \frac{\partial M}{\partial \mathrm{y}}\right) dA \tag{32}$$

$$\oint_C \frac{1}{2}xdy - \frac{1}{2}ydx = A \tag{33}$$

$$A = \frac{1}{2}\oint_C Mdx - Ndy \tag{34}$$

Each of the sub-intervals within the parametrized interval has a length of Δs. So, Δs means difference on x and y axis. When the function is $f(x_i, y_i)$, the defined interval's area equals to:

$$A = \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i \tag{35}$$

This equation resembles the Riemann Sum since each x and y value may be computed using the function f. Difference on x and y values can be notated as dx and dy. Here dx and dy is can be rewritten in coordinate form:

$$dx = \Delta x = x_{i+1} - x_i \tag{36}$$

$$dy = \Delta y = y_{i+1} - y_i \tag{37}$$

To have a surface area formula, Equation (34) is rewritten:

$$A = \frac{1}{2}\oint_C x_i(y_{i+1} - y_i) - y_i\,(x_{i+1} - x_i) \tag{38}$$

Again using the rearranging procedure Equation (34) to (35), Equation (38) is rewritten:

$$A = \frac{1}{2}\sum_{i=1}^{n} x_i y_{i+1} - y_i x_{i+1} \tag{39}$$

Now, the area and dimensions of Great Britain will be calculated using the surface area and perimeter relationship. Pixel coordinates on the page taken from Adobe Photoshop are used to calculate. The coordinates of points on two maps that are given in Table 5 and Table 6 are shown here:

Table 5 *Coordinates of straight-line nodes that are shown third iteration of the Hausdorff-Besicovitch method*

x	y	*A* pixel^2	x	y	*A* pixel^2	x	y	*A* pixel^2
135	516		113	218	4549	310	335	4267.5
182	515	-12193.5	117	160	-3713	323	377	4332.5
215	476	-12046.5	112	121	-1881.5	365	408	-2910.5
165	500	14480	132	95	-2666	382	454	4927
132	473	6022.5	86	86	1591	348	490	14594
148	431	-6556	142	44	-4214	348	533	7482
139	384	-1538.5	184	23	-2415	309	545	12481.5
189	378	-10017	185	69	4220.5	259	560	15942.5
200	334	-6237	226	83	-119.5	210	550	12425
183	291	-1461	250	125	3750	160	555	14275
138	264	4077	223	166	6812.5	62	582	29355
144	215	-4173	244	213	3497.5	∑	98757.5 pixel^2	
134	178	-1589	267	261	3406.5	∑	211813.2 km^2	

Great Britain has a surface area of 98757.5 pixel^2 when all of the data in Table 5 are added up. It is 211813.2 km^2 since each pixel is 0.91 km when converted to km^2.

Table 6 *Coordinates of straight-line nodes that are shown second iteration of the Hausdorff-Besicovitch method.*

x	y	*A* pixel^2	x	y	*A* pixel^2	x	y	*A* pixel^2
142	533	0	203	40	-7465.5	377	547	22010
160	446	-10974	274	112	5888	279	572	31515.5
211	360	-18253	252	211	14795	179	566	27763
156	283	1776.5	292	300	6994	63	593	35244.5
145	181	-6399.5	331	378	5538	∑	105413.5 pixel^2	
119	97	-3737	392	452	718	∑	226088.9km^2	

Great Britain has a surface area of 105413.5 pixel^2 when all of the data in Table 6 are added up. It is 226088.9 km^2 since each pixel is 0.91 km when converted to km^2.

To determine the percent error for this method Equation (29) is used:

$$D = \frac{2\log\left(\frac{2000 * 100}{1625 * 50}\right)}{\log\left(\frac{211813.2 * 10000}{226088.9 * 2500}\right)} \quad (40)$$

$$D = 1.3637218 \quad (41)$$

$$\delta = \frac{|1.3637218 - 1.25|}{|1.25|} \times 100 \quad (42)$$

$$\delta = \%9.1 \quad (43)$$

4. Determination of Antarctic Coastline Length as a Case Study

However, the obtained value can be utilized for comparison at a constant scale. As a student concerned with the effects of global climate change, I wish to help scientists in understanding the effects of climate change in Antarctica through the study of field geography. By the year 2100, scientists expect Antarctica's sea level to rise by 25 cm. Scientists believe that this problem won't just affect the Antarctic; it will affect many other places as well (Jenouvrier 2021). I hope that the change in coastal length and surface area over time will be helpful in helping scientists understand the challenges we will encounter. In this section, two different Antarctic maps will be chosen as a case study and studied using the methods stated above in order to obtain an approximation for the coastline length and surface area change during glacial cycle.

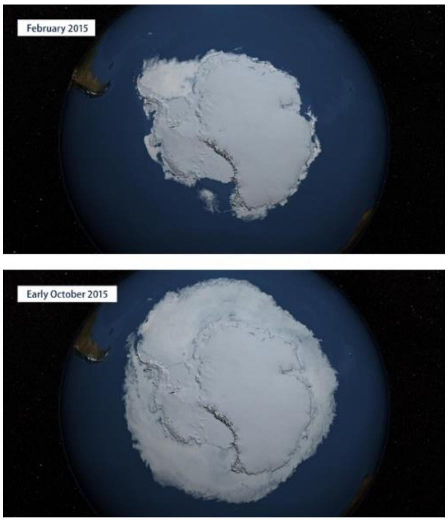


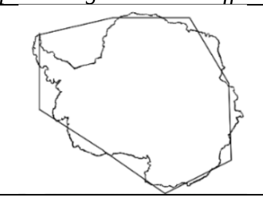
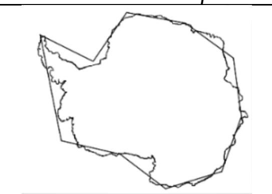
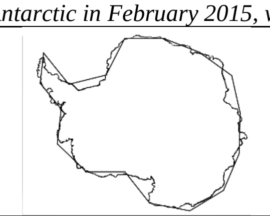
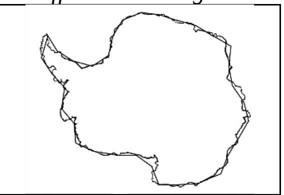
Figure 3 Maps of Antarctic in February 2015 and in Early October 2015 (Viñas, 2015).

Two examples are shown Figure 3, one from early October 2015 and one from February 2015. Maximum sea ice extent in Antarctica surprisingly broke the run in 2015. Different methods will be taken to comprehend how the length of the coastline and surface area varied during this time (US EPA, 2016). The fractal dimension value will regulate several methods. Because a coastline must always have the same fractal dimension, its length and surface area might vary depending on the method used.

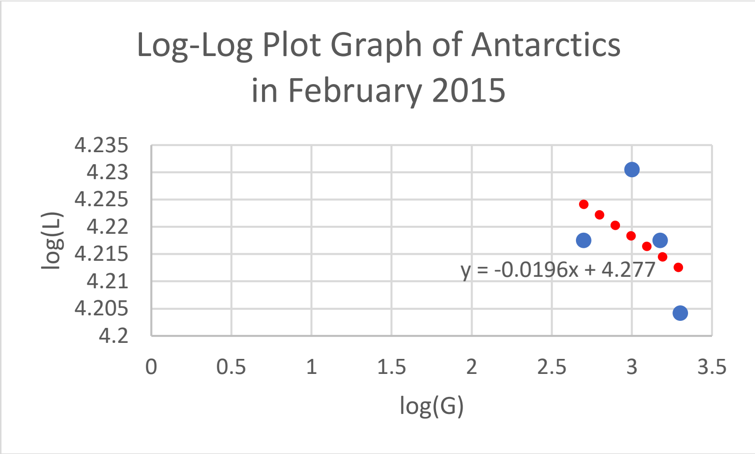
4.1 The Hausdorff-Besicovitch Method (D_H)

The slope of the resulting graph will be used to estimate the fractal dimension. The sketched straight-line segments in Table 7 on the Antarctic map in February 2015 got smaller. The coastline of Antarctica is around 1700 kilometers long, as seen in Table 7.

Table 7 *Map showing the Hausdorff-Besicovitch Dimension for the Antarctic in February 2015, with different unit lengths.*

				
L (km)	16000 km	16500 km	17000 miles	16500 kilometer
G (km)	2000 km	1500 km	1000 km	500 kilometer
n	8	11	17	33
log (L)	4.204119983	4.217483944	4.230448921	4.217483944
log (G)	3.301029996	3.176091259	3	2.698970004

Such as in the Great Britain example, just four iterations are utilized. A log log plot graph of the gathered data is then constructed, and the gradient of the data is determined by the trendline.



Graph 3 *Antarctic's the Hausdorff-Besicovitch fractal dimension as a log-log plot graph in February 2015.*

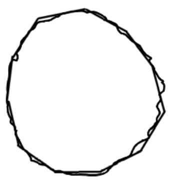
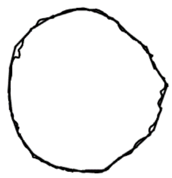
The gradient of the trendline with the plotted data is "-0.0196," as can be seen in Graph 3. The following equation is used to convert this value into Mandelbrot dimension:

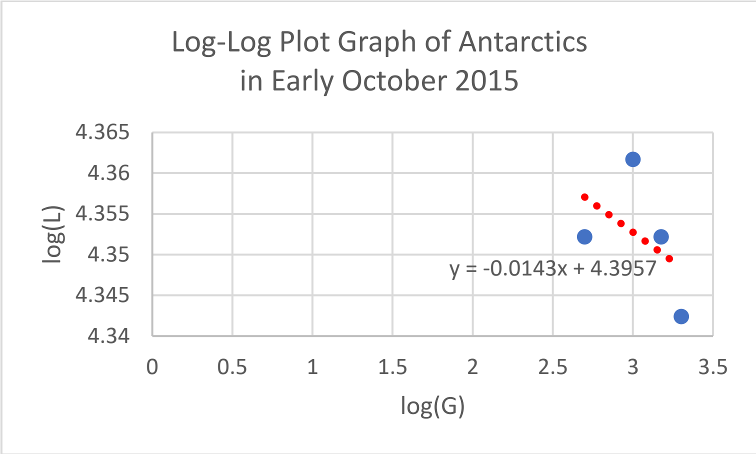
$$D_M = 1 + |(-0.0196)| \quad (44)$$

$$D_M = 1.0196 \quad (45)$$

In early October 2015, the Antarctic map is created using the same method. The coastline of Antarctica measures around 2300 kilometers long, according to Table 8.

Table 8 *Map showing the Hausdorff-Besicovitch Dimension for the Antarctic in early October 2015, with different unit lengths.*

				
L (km)	22000 km	22500 km	23000 km	22500 km
G (km)	2000 km	1500 km	1000 km	500 km
n	11	15	23	45
$\log(L)$	4.342422681	4.352182518	4.361727836	4.33243846
$\log(G)$	3.301029996	3.176091259	3	2.698970004



Graph 4 *Antarctic's the Hausdorff-Besicovitch fractal dimension as a log-log plot graph in early October 2015.*

$$D_M = 1 + |(-0.0143)| \tag{46}$$

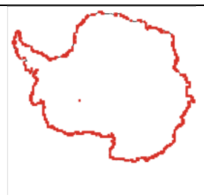
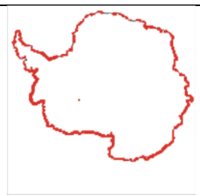
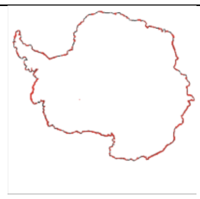
$$D_M = 1.0143 \tag{47}$$

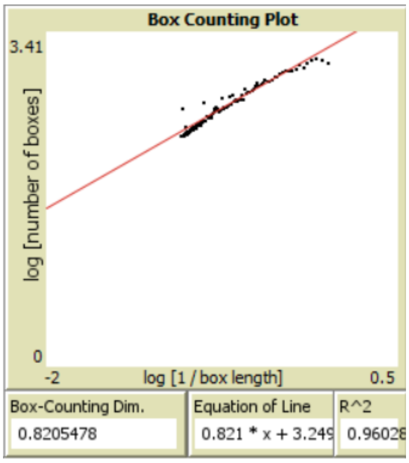
Early October 2015's map of the coastline has a smoother coastline, and as would be predicted, it has a smaller fractal dimension than the initial map (-0.05).

4.2 The Minkowski-Bouligan Method (D_B)

As previously stated Equation (18), is going to be used to determine the length of the coastline of Antarctic.

Table 9 *To determine Antarctic's fractal dimension in February 2015, the Minkowski-Bouligand method is used with the NetLogo software.*

				
N	532	786	1039	1338
ε	4	3	2	1
Iteration #	30	20	10	1

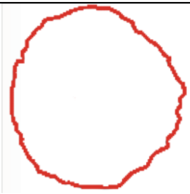
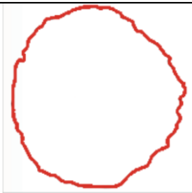
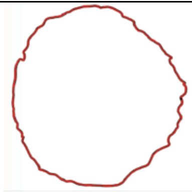
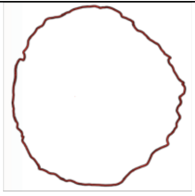


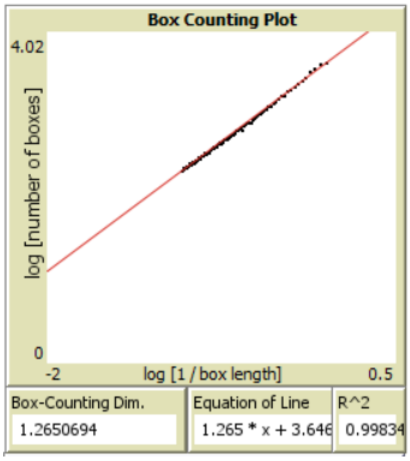
100 iteration of Minkowski-Bouiligand method gives the Graph 5 which has gradient equals to about "0.82".

Graph 5 *Obtained from the NetLogo program the Minkowski-Bouligand log-log plot graph and its gradient value (fractal dimension) for Antarctic in February 2015.*

In early October 2015, the Antarctic map is created using the same method.

Table 10 *To determine Antarctic's fractal dimension in early October 2015, the Minkowski-Bouligand method is used with the NetLogo software.*

				
N	675	1086	1792	4460
ε	4	3	2	1
Iteration #	30	20	10	1



The second map's fractal dimension is shown by the number "1.27" in Graph 6. This is a surprising outcome given that the second map appears to be less complex than the first one when using the Hausdorff-Besicovitch approach.

Graph 6 *Obtained from the NetLogo program the Minkowski-Bouligand log-log plot graph and its gradient value (fractal dimension) for Antarctic in early October 2015.*

4.3 The Surface Area to Perimeter Ratio of Fractal Structures

In this section, using the previously stated ratio; Antarctic's surface area and fractal dimension will be calculated.

Table 11 *Coordinates of straight-line nodes that are shown in the third iteration of the Hausdorff-Besicovitch method.*

x	y	Apixel^2	x	y	*A* pixel^2	x	y	*A* pixel^2
78	94		755	175	20487.5	371	526	2105
168	206	138	785	314	49847.5	229	526	37346
305	174	-16799	783	460	57619	127	424	15147
370	40	-26090	726	593	65179.5	140	298	-10757
509	41	-2595	604	667	63035	65	205	4665
644	95	10975.5	458	638	39933	∑	310237 pixel^2	
						∑	13719455.7 km^2	

Antarctic in February 2015 has a surface area of 310237 pixel^2 when all of the data in Table 11 are added up. It is 13719455.7 km^2 since each pixel is 6.65 km when converted to km^2.

Table 12 Coordinates of straight-line nodes that are shown in the second iteration of the Hausdorff-Besicovitch method.

x	y	*A* pixel^2	x	y	*A* pixel^2	x	y	*A* pixel^2
66	107		763	191	31929	352	537	22487.5
257	203	-7050.5	793	402	77631.5	143	490	47844.5
377	27	-34796	721	603	94168.5	106	311	-3733.5
590	64	4099	519	664	82893.5	∑	315473.5 pixel^2	
						∑	13783699.7 km^2	

Antarctic in February 2015 has a surface area of 315473.5 pixel^2 when all of the data in Table 12 are added up. It is 13783699.7 km^2 since each pixel is 6.61 km when converted to km^2. Equation (29) is applied to acquired data:

$$D = \frac{2\log\left(\frac{17000 * 1500}{16500 * 1000}\right)}{\log\left(\frac{13719455.7 * 2250000}{13783699.7 * 1000000}\right)} \quad (48)$$

$$D = 1.079847459 \quad (49)$$

Table 13 Coordinates of straight-line nodes that are shown in the third iteration of the Hausdorff-Besicovitch method

x	y	*A* pixel^2	x	y	*A* pixel^2	x	y	*A* pixel^2
82	156		492	159	5445	297	550	19145.5
134	102	-6270	517	217	12280.5	218	534	19349
202	64	-6014	559	277	10953	148	500	14984
278	39	-4957	540	349	22755.5	104	50	-22300
332	44	-358	518	420	23009	72	374	17648
374	67	2894	461	480	27510	63	298	-1053
410	88	2721	412	504	17292	68	230	-2887
446	122	5386	365	547	20702	∑	178235.5 pixel^2	
						∑	14372982 km^2	

Antarctic in early October 2015 has a surface area of 178235.5 pixel^2 when all of the data in Table 13 are added up. It is 14372982 km^2 since each pixel is 8.98 km when converted to km^2.

Table 14 Coordinates of straight-line nodes that are shown in the second iteration of the Hausdorff-Besicovitch method

x	y	*A* pixel^2	x	y	*A* pixel^2	x	y	*A* pixel^2
189	63		628	449	47781	69	403	5025.5
310	39	-6079.5	546	561	53577	63	271	-3345
418	42	-1641	440	640	51300	112	158	-10199
500	127	16043	319	645	39820			
608	209	13642	195	605	33610	∑	284930.5 pixel^2	
666	324	28899	115	526	16497.5	∑	14688851.1 km^2	

Antarctic in early October 2015 has a surface area of 284930.5 pixel^2 when all of the data in Table 14 are added up. It is 14688851.1 km^2 since each pixel is 7.18 km when converted to km^2. Then, the same procedure is applied:

$$D = \frac{2\log\left(\frac{23000 * 1500}{22500 * 1000}\right)}{\log\left(\frac{14372982 * 2250000}{14688851.1 * 1000000}\right)} \quad (50)$$

$$D = 1.083245176 \quad (51)$$

5. Conclusion

The change in Antarctic surface area and perimeter in the 2015 glacial cycle is computed in accordance with the goals of this exploration considering fractal dimension as a control parameter. In this study, methods are used on a Great Britain example to first understand fractal dimension finding methods. The fractal dimension of Great Britain has already been reported by a number of sources, and the reliability of each method is determined by comparing the obtained results to the existing literature. Then, using Antarctica's two maps of the 2015 glacial cycle as a case study, the aforementioned methods are put into practice.

Table 15 *Methodologies and their percent error.*

Methodology	Percent Error (%)
The Hausdorff-Besicovitch Fractal Dimension	%1.06
The Minkowski-Bouligand Fractal Dimension	%9.89
Area-Perimeter Relationship Fractal Dimension	%9.10
The Green’s Theorem Surface Area Calculation	%7.12

To obtain more precise results in the fractal dimension, the Hausdorff-Besicovitch method is used for Antarctic, taking into account these percent error values.

Table 16 *Acquired fractal dimension value and surface area value for Antarctic’s case study.*

Values for Antarctic	in February	in Early October
Fractal Dimension (The Hausdorff-Besicovitch Method)	1.02	1.01
Surface Area (The Green’s Theorem)	13750000 km^2	14530000 km^2
Coastline Length (The Hausdorff-Besicovitch Method)	16500 km	22500 km

Given that the complexity of the curves on the second map is less than that on the first, the fall in fractal dimension is to be expected. The difference in surface area is approximately 7800 km^2. The difference in coastline length is approximately 6000 km.

There are several ways to get a more accurate result for each method. Just four iterations are used in the Hausdorff-Besicovitch method, but this number can be increased to produce results that are more precise. The Miskowski-Besicovitch method makes use of an outdated NetLogo-based software program. As demonstrated in Antarctic's case study, some integration issues influenced the results. Although the software produced this result, a value for fractal dimension below 1 is not what was anticipated. The area is determined using the Green's Theorem, and the area calculation would be more accurate if the number of iterations were increased. The EPSG Geodetic Parameter Dataset 3857 is used in this investigation for all maps. If different datasets were used, methodologies would yield different results. Hence this parameter is used to create a standard for coordinate transformation and measurement units in maps (Cain, 2013).

This investigation gave me insight into the use of mathematics in fractal geometry. The fact that nature always has a mathematical explanation, as I already stated in the introduction, intrigued me greatly. Now I understand what Pythogoras meant when he said, "There is geometry in the humming of the strings. There is music in the spacing of the spheres.” As a result of this study, I am also able to appreciate the beauty of mathematics; fractal geometry is absolutely stunning. Throughout this study, I had time to consider various, more scientific ways to tackle the effects of climate change. This mathematical modeling of the Antarctic that I did can be applied to various regions, islands, and continents for further research.

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